***Provisional translation***

**One. Market risk estimation (calculation)**

* 1. **Basic concepts**

In banking operation are more exposed to market, credit, operational and solvency risks are relative to other types of risks. Aside from these, additional risks such as reputational, legal, country and other risks influence banks’ operation that are often difficult to estimate. Although credit risk is arguably the most important source of risks, its counterparts – market and operational risks more likely expose a bank to default and loss events than other types of risk which makes it essential to estimate and control them. While the first accord promulgated by the Basel committee of the Banking supervision in 1998 mainly focused on estimating the potential credit risks that banks are exposed to during predefined horizon and the bank’s ability to cover the risk with its capital, the following accord otherwise known as Basel II, along with credit risk, also added market and the operational risks by presenting the estimation methods and bank’s ability to absorb the losses derived from these risks with its preserved capital.

For the pricing of financial instruments owned by a bank, the core inputs are product price, interest rate, their standard deviation and maturity. In other words, the financial instruments or product pricing can be expressed as a function (or pricing function) of those factors such as price and maturity. The valuation of the instruments is largely dependent on the fluctuations on these factors, as a result of which a bank may bear loss. Simply put, the market risk is “**the potential that bank may bear due to change in the value of a financial instrument it holds caused by the likely fluctuation in the factors that determine the value of a financial instrument**”.

As statistical science and the probability theory suggest that it is better to apply relative measures than absolute measures in estimation procedures, the market risk estimation also uses relative measures. Thus, the price change is expressed as the relative measures in market risk estimation and the value change in financial instruments can be derived through the use of pricing function. Relative measures are presented in two major ways, of which:

1. Relative change - is the ratio change in the nominal measure of the factors that influence financial instrument’s value or a valuation to the sum of the nominal values.
2. Log relative change – the rationale for choosing this measure is based on the assumption of continuously compounding interest rates and is calculated as the ratio of current nominal value to the previous nominal value (a relative measure of sorts) and then taking the natural logarithm – LN from the ratio.

Depending on the price, if we write the financial instruments valuation as  (where  and the variables  are price and maturity (In a specified case, a one dimensional vector where it is the function of one variable.), then in order to measure credit risk we need to estimate the effect of derivatives of the function with respect to *X* on the financial instrument valuation.

However to avoid the complexity of applying the function as is, it is often represented or approximated through the polynomial method through Taylor function (series) which approximates the interrelationship between the effect of derivatives of factors concerned, the relative change in the financial instrument’s valuation and the relative change in the factors with minor errors. The univariate function  dependent on factors can be expressed with Taylor series around the point  as follows:

Where  and if we denote the relative change in value of the financial instrument and relative change in the factors as  respectively then the expression becomes. If the second or the higher order derivatives of the function are deemed of small influence then  becomes the scalar multiplier.

Above form is applied when only one factor (simplified version) is used to value the financial instruments. For the valuation of more complex financial instruments – option, for example, several factors such as maturity, value of the underlying asset, its standard deviation. In this case, the multivariate version of the function  should be used. For the multivariate function with two variables around the points, Taylor series takes the following form:



In most cases, though the effect of the derivatives of higher orders than that of the first of the valuation function is small, for many complex financial instruments such as the option, there is a considerable effect by the second order derivative of the valuation function. As shown above, if the effect of second or the higher order derivative of the function is zero or infinitesimally small, the relationship between the relative change in value of the financial instrument and that in the selected factors can be expressed with the linear approximation of Taylor series. If there is a noticeable effect from the second or the higher order derivatives of the function, the relationship becomes non-linear. Since the effect of the derivatives of orders higher than that of the second, usually the effects of first and second derivatives of the function are applied. If the option valuation function is of the form -  (where -is the value of underlying asset of the option, -strike price, -maturity, -standard deviation of the underlying), it is expressed as the following multivariate version of the Taylor series:

Where  . If we denote the relative change in value of the financial instrument and relative change in the factors as  respectively then the expression becomes  or  . This formula can be rewritten in abbreviated version as. In other words, if there is an effect from the second order derivative of the valuation function, the relationship is of the non-linear or the quadratic form between the relative change in value of the financial instrument and relative change in the factors - .

Hereafter, the below terminologies will have the following meanings:

* ***Relative change of the valuation*** *–* means the value that the valuation function takes or the relative change in the financial instrument.
* ***Relative change in real variable***[[1]](#footnote-2)- is the variable of the valuation function or the relative change of the factors (in short, the relative change of the main variable). Furthermore, relative change in real variable should mean main factor of the valuation function - the price change.
* ***Delta effect*** – is the multiplication other than the change of the main variable of the summation derived by decomposing the valuation function into Taylor series containing the first order derivative of the abbreviated form of the relative changes in both the valuation and its main factors.
* ***Gamma effect*** - is the multiplication other than the relative changes in both changes in main factors of the summation containing the second order derivative of the expression shown in the third explanation (above).
* ***Portfolio*** – is financial instruments held for trading purpose.
* ***Portfolio valuation*** *–* is the sum of the valuation of the financial instruments held for trading purpose.
* ***Linear financial instrument*** – is the financial instrument where the effect of the derivatives higher than that of first of the valuation function approximated with the Taylor series is virtually nonexistent or zero.
* ***Non-linear financial instrument*** - is the financial instrument where the effect of the derivatives higher than that of first of the valuation function approximated with the Taylor series is quantifiable (the valuation of derivatives is dependent on the valuation of the simple financial instrument).

In risk methodologies, the notations below are applied with the following meaning[[2]](#footnote-3), of which:

- is a value of the real variable of the financial instrument  at time .

- is a nominal value or an amount of the financial instrument  at time .

- is a value of the portfolio value at time 

- is a relative change of 

- is a relative change of 

- is a relative change of 

-is a standard deviation or a volatility of variable (- variance of )

-is a covariance between and 

-is a standard deviation of the portfolio value at time 

- is the amount[[3]](#footnote-4) of the standard normal distribution at probability.

- is a normal density function with zero mean and,  standard deviation

- is a standard normal density function with zero mean and unit standard deviation

- is a measure of dependence or a correlation coefficient between  and 

- is a correlation matrix

-is a weight of a financial instrument in the value of the portfolio

-is a covariance matrix of a portfolio at time 

- is the first order derivative of a valuation function of the financial instrument  with respect to 

- is the summation containing the first order derivative of the value of a financial instrument  in the Taylor series with respect to 

- is the second order derivative of the valuation function of financial instrument  with respect to 

- is the summation containing the second order derivative of the value of a financial instrument  in the Taylor series with respect to 

- is a first order derivative of the valuation function of financial derivative with respect to time variable

- is the summation containing the derivative of the value of a financial instrument  in the Taylor series with respect to time.

The table below shows classifications of commonly used financial instruments and their degree of dependence on their variables

|  |  |  |  |
| --- | --- | --- | --- |
| № | Financial instruments | | Real variables |
| 1. | ***Linear financial instruments*** | | |
| 1.1. | *Linear Plain vanilla financial instruments* | | |
|  |  | Bond | Bond price |
|  |  | Stock | Specific market index |
|  |  | Foreign currency trading | Foreign Exchange rate |
|  |  | Commodity | Commodity price |
|  |  | Interest rate swap | Swap price |
| 1.2. | *Linear derivative instruments* | | |
|  |  | Variable interest rate bond | Money market price |
|  |  | Foreign currency forward | Foreign exchange rate |
|  |  | Forward interest rate contract | Money market price |
|  |  | Currency swap | Swap price/foreign exchange rate |
| 2. | *Non-Linear derivative instruments* | | |
|  |  | Stock option | Stock price |
|  |  | Bond option | Bond price |
|  |  | Foreign currency option | Foreign exchange rate |

For any market risk estimation methodology, the real variables are mostly assumed to have a normal distribution. Accordingly, the relative change of a real variable also follows normal distribution and has the advantage of zero mean or an expected value. In other words, . Hence real variable is normalized or converted to the standard normal distribution via  (random variable with standard normal distribution, value with normal distribution, standard deviation of ). (If the expected value is a non-zero , then via formula  can be converted into the standard normal distribution and the computation becomes difficult of the portfolio with multiple financial instruments.) Thus, with significance level -, the value that can take can be estimate by taking inverse of the standard normal distribution and with denote them as  or , then the value of with probability can be derived from the normalized standard normal function as . Significance level -has the range between 1% and 10% and mostly takes the values of 1%, 5% and is used as the external dimension in this methodology.

* + 1. **Linear financial instrument**

If -is the relative changes of real variables of a linear financial instrument, then the relative change of a financial instrument valuation in Taylor approximation has a normal distribution where the maximum change or the risk that financial instrument value can take at  significance level after unit time horizon (at time ) is  (where -is value of the financial instrument).

Although, usually there is a 1:1 relationship between the relative changes of linear financial instruments and their real variables, in some cases where the continuously compounding method is taken into account in the valuation, the delta value – that is the first order derivative of the valuation function is no longer a constant meaning the second order derivative has a nonzero value. **In this case, the errors from the risk estimation with the method above are considerable.** Hence, if the continuously compounding method is used, then the non-linear methodology should be applied for risk estimation.

Bank not only estimates the individual risks of each financial instrument they hold, but also assess its portfolio risk. To do so, weights of each financial instrument in the overall portfolio (where - weight of financial instrument  in the portfolio, - a column vector where the members are ) and the effect of the interrelationship between real variables of financial instruments in the portfolio need to be incorporated into the methodology. By including these effects, there is a possibility that the final portfolio risk is less than (in some cases more than) the sum of the risks of individual financial instruments in the portfolio. Thus, when evaluating the portfolio risk, the correlation matrix  (a symmetric matrix containing the members as the correlation coefficient between financial instruments -  ) or its non-normalized form namely covariance matrix  (a symmetric matrix containing the members as the correlation coefficient between financial instruments -  ) need to be calculated.

* + 1. **Non-linear financial instrument**

If a financial instrument is non-linear, the relative change of its valuation, depending on the second order derivative of the valuation function, not always follows a normal distribution similar to the relative change in its real variables. Thus, it is important to find the distribution that the relative change in the valuation is following from which the maximum value of the relative change at a certain significance level can be calculated.

With a portfolio that bank holds containing the non-linear financial instruments, a calculation of a portfolio risk with such structure is different with the one that has linear financial instruments entirely in it. In other words, a gamma effect needs to be included in the portfolio risk estimation.

* 1. **Simple risk estimation method**
     1. **Historical simulation**

This method underlies the idea that the distribution for the relative change of values at time  has the same or similar to the distribution of observations at previous  periods (that is,). For financial instrument  the set of past observations of relative changes of values is defined as follows:



With  confidence level, the maximum amount that the relative change of a financial instrument  takes can be calculated by forming the frequency that falls into predetermined interval. The formula for the calculation is:



For portfolio risk estimation using the above method, just as shown in the case for the single financial instrument, the observations of relative changes of values in the previous periods are taken as the direct set and unlike other risk estimation methods this one does not require estimating correlation among financial instruments. To derive the set of the observations for previous periods, the sum of relative changes of individual financial instrument value weighted with its corresponding share in the overall portfolio value for each period following the below formula.



From there, we can estimate the maximum value of a relative change in a portfolio at the -confidence level at time  as:



The advantage for this method is that it does not apply parameters or it does not require parameter estimation, valuation of financial instrument and the correlation among its variables or the valuation function. However, it shows the disadvantage of choosing the size of -observation amount in that if it is too high then the effect of observations at previous periods would be understated causing the precision risk stable. On the other hand, if -is specified too low, the precision of an estimation may be compromised due to the undermining the effect from the observation of last periods. In most cases, for this method, *m* is chosen around 250-1000.

This method assumes that both the relative changes of individual financial instrument and the overall portfolio follow normal distribution and in that sense is similar to the Delta-Normal method.

* + 1. **Delta normal method**

This method assumes the effect of derivatives of higher orders except the first order is zero and thus only applied to estimate the risks of linear financial instrument or the portfolio containing them. Its method is the same as converting normally distributed random variables into standard normal distribution. Thus, the relative change of a chosen variable *Xi,t* is maximum change or a risk of a financial instrument at time *t+1* with *p* confidence level is estimated as follow:



A product of a last two members of the right hand side of the expression is the relative change of main variable in unit time or a maximum value *Xi,t+1* at time *t+1* with *p* confidence level. Since the method is used for the valuation of linear financial instruments, standard deviations of valuation and the variables are equal and thus product of *Xi,t+1* and *Yi,t* makes the maximum change of *Yi,t+1* with *p* confidence level or the risk.

If a portfolio contains solely the linear financial instruments, the method for a portfolio takes the following form:



As shown for a valuation of single financial instrument of converting normal distribution into a standard normal, the same principle applies for a portfolio as follows:

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The above form of risk estimation is used only when there is a ground that relative change of a portfolio follows conditional normal distribution. In other words, it is assumed that a relative change of a portfolio has a conditional normal distribution.

* 1. **Dynamic risk estimation methodology**
     1. **Exponential Weighted Moving average methodology**

In recent methodology, the standard deviation at time *t+1* of financial instruments constituting the portfolio was estimated with equal weights for each observation period. In other words, the risk of time *t+1* is assessed at time *t* with simple mathematic or an arithmetic average and when the option for the number of observation of previous periods *m* is set high, there is an equal treatment to the information collected at time *t+1-m* and *t-1*. This makes it the main cause of a large error of risk estimation or assessment from the actual estimate.

To avoid this issue, the weighted average method which treats deviation at each period differently in that the deviation at the later period is assigned higher weight to that of the earlier periods. One of these methods is the Exponentially Weighted Moving Average[[4]](#footnote-5) method where the deviation at time *t+1* is calculated as follows:

 Where - is a positive real number less than 1.

This form is the special case of GARCH method where  and -is calculated using Maximum Likelihood Method. For instance, JPMorgan sets at 0.94. By decomposing the second component of a right-hand side of above expression, it takes the following recurrent form:



In other words, it takes the form of a geometric progression in which deviations of main variables are weighted with the multiplier less than one. The main difference from the prior method is that this method determines the deviation at time *t+1* with the deviation of main variables and their relatives changes at time *t*.

By incorporating the covariance matrix of main variables of financial instruments into the method we get:

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Multipliers  and  can be taken as scalars into the above expression. From there, portfolio deviation and portfolio risk with the confidence level  can be calculated as follows.

, .

* + 1. **GARCH model**

Although the choice of risk estimation method mostly depends on the characteristics of relative changes of the main variables, GARCH methods are widely applied. Delta-normal method assumes the independence between standard deviations of main variables for previous periods. This type of method is called homoscedastic method. However in some cases, volatilities of relative changes of main variables are correlated with those of previous periods, an alternative method that better reflects this effect is needed to estimate the volatility of relative changes in main variables more precisely. This type of method which calculates the time dependency is called heteroscedastic method. Despite GARCH (Generalized autoregressive conditionally heteroscedastic) method better shows the characteristics of relative changes in the main variables, compared to abovementioned methods, it has disadvantages that multiple numbers of parameters estimations are required. For single financial instrument, a GARCH model takes the following form:

This form of expression is noted in short as GARCH(p,q), where its simplest form GARCH(1,1) has the expression . (Where ) Weighted average method is the special case of GARCH (1,1) where . These parameters are estimated using Maximum likelihood method. Since the main variables have normal distribution, maximum likelihood function also has the normal distribution. For time *t*, a probability that is a maximum likelihood equals the following:



From here, a maximum likelihood during the overall periods is the product of probabilities of each period.

 Or taking natural logarithm we get: . Here  is a parameter estimate and the objective is to find the values of  in the summation (in the above summation, only these parameters are used as variables) function lnL containing individual likelihoods of n observations that maximizes the same function. When parameters are estimated, a risk of a given financial instrument at time *t+1* at *p* confidence level is calculated as:



For GARCH method, in order to simplify the correlations between main variables for a portfolio numerous alternative calculations are applied. Here, three extended forms of GARCH method to estimate the portfolio risks are explained.

* + - 1. ***VECH model***

For a portfolio, aside from the variances of main inputs their covariances also need to be estimated. In order for the effects of all other variances and covariances integrated to the estimation of each given variance and a covariance, a vector or column matrix expression containing not only the variances of main variances but also their covariances is used for extended GARCH model. The method of this approach is called VECH (where, vec - means vector, and H indicates covariance matrix for short) method.



Where , and  are  dimensional, symmetric matrix, - is vector method that extracts the upper right members of a diagonal matrix. (In some cases, instead of using  matrix, keeping in mind the product of the upper or lower triangular matrix and their transposed matrix results symmetric matrix, with  form or upper and lower triangular matrix are used). Based on that, total portfolio standard deviation is estimated as follows:



* + - 1. ***Diagonal VECH model***

For a portfolio with two financial instruments, the VECH model estimates overall 21 parameters using maximum likelihood method. In actuality, a portfolio consists of greater numbers of financial instruments and requires greater numbers of parameters estimation complicating the calculation. For banks that invested in large numbers of financial instruments with the increased number of parameters to be estimated requiring the sophisticated software programs which incur high cost. Thus, VECH model in some case is simplified with diagonalizable VECH method. In this method, Matrix *A* and *B* are expressed in diagonal matrices and has the same expression as *VECH* model.



As *A* and B are diagonal matrices, members of a covariance matrix at time *t+1* are as follows:

. Here .

From there, a portfolio standard deviation is estimated as:



* + - 1. ***BEKK model***

Whereas in VECH model variances and covariances are put together in vector form, in this model covariances and variances are expressed in matrix form. Model has the following form:

. Here , and  are  dimensional matrix.

In some cases, in place of matrix, - formed upper and lower triangular matrix is used. Based on that, portfolio standard deviation at time *t+1* becomes:



* + 1. **ARCH**

In the beginning, due to the assumption of independence in terms of time between variances and covariances of the main variables, Historical simulation and Delta Normal methods were used for market risk estimation. Then, the alternative dynamic methods that incorporate time dependency between covariances and variances were required and one of the first (for market risk) method was ARCH (autoregressive conditionally heteroscedastic) model. The general form of this method is:



If above form is denoted ARCH(p) the case of  ARCH(1) has the following form.



GARCH model was established based on this premise and based on the above expression the effect of main variables or the leverage effect was excluded from this method. However, parameters need to be estimated with Maximum likelihood method.

A portfolio deviation is estimated using method similar to GARCH method and in this case a summation containing the main variable is not applied. However in practice, the extended version of ARCH model is not applied.

* 1. **Risk estimation of non-linear financial instruments (option)**

For non-linear financial instruments such as options, the effect of second order derivative of a valuation function is more than zero.

In order to estimate the risk of non-linear financial instruments, let us take the example of option function. Suppose a valuation of a simple stock option follows Black-Scholes formula with  form. Here,  is a main variable at time or in this case a spot price of a stock, *K* is option strike price, *t* is time to settle (final) the option transaction (mostly expressed in years),  is a risk-free return during the settlement of an option,  is standard deviation of a stock or logarithms of main variables price during the time of the option (final deadline to settle the transaction). We saw with Taylor series, if higher than second order derivatives of this or multivariate valuation function is zero, then the following expression is used:



* + 1. **Delta-Gamma method**

In the section of simple risk estimation, we saw how to apply Delta-Gamma method when there was a linear relationship between relative changes in the prices of financial instruments and in main variables or price of a underlying asset. From there, it is inferred to estimate options risk, effects of delta, gamma and theta need to be estimated. If the valuation function is known, its effects of derivatives or a value can be estimated using values of main variables at time *t.*

The effects of gamma and theta in the distribution of relative changes of option values using the comparison between the relative change of option values and moments of relative changes in main variables can be shown the following way. The relative changes of main variables  are normally distributed with  mean,  standard deviation. The table below shows , and first 4 moments of portfolio valuation including mean, deviation, scenes and kurtosis.

|  |  |  |  |
| --- | --- | --- | --- |
| **Statistical Parameters** | **Variables** | **Option valuation** | **Portfolio valuation[[5]](#footnote-6)** |
| Relative change |  |  |  |
| Mean |  |  |  |
| Deviation |  |  |  |
| Skewness |  |  |  |
| Kurtosis |  |  |  |

Three conclusions are drawn from the table above. Of which:

1. Although mean of the variables are zero, if gamma and theta are non-zero then the relative changes of option values are non-zero. And the sign of the mean of relative changes of option values depends on the signs of theta and gamma irrespective of whether the option is short or long.
2. The deviation of relative changes of an option differs from the deviation of relative changes of variables by  multiplier. The sign of skewness of a distribution defines whether the option is long or short. If option is short  is negative and so is the skewness indicator .
3. Since for skewness and kurtosis,  participates with even power, the signs of skewness and kurtosis are no different than that of variables.
   * + 1. **Johnson interpolation**

Based on first 4 moments of the relative changes of option valuation, a distribution with similar 4 moments is sought. In other words, so-called Johnson interpolation is used to calibrate the distribution moments of relative changes of a valuation to that of the options. Moment calibration means to adjust and find the parameters of a given distribution whose shape is known and its mean, deviation, skewness and kurtosis to those of relative changes of valuation. This leads to the problem to define interpolation that shows relative changes of option valuation from the relative changes of variables with standard normal distribution. Here the general form of Johnson interpolation is shown below:

. Here  is monotonous function,  are the parameters defined by the first 4 moments of relative changes of option valuation. This interpolation factor generally takes the 3 forms:

- With one-sided limit ()

-limited from both sides ()

- Not limited (also denoted for short)

Where,  is the measure with standard normal distribution and if this measure with univariate standard normal distribution is inserted into the multivariate standard normal distribution using the above 3 expressions we get the density function of  which takes the following form:



Where the derivative of function  is calculated as shown below:

- When limited by one side

-when limited from both sides

- Not limited

From here the first 4 moments of  should equal the first 4 moments of in the table above. Thus, we get overall 4 system equations with *a, b, c, d* parameters and theoretically the unique solution from the system can be derived. (However, the moments are in integral form of density function and the integral of Gaussian function is not represented by an elementary function, a specified methods for  parameters calculation is used. The most widely used estimation methods are Hill, Hill and Holder’s algorithm[[6]](#footnote-7)). After the parameter estimates are calculated, the estimates of  at *p* confidence level are inserted in above expression using the Johnson interpolation we get the estimates of  at  confidence level that is. From here, an amount of risk of a non-linear financial instrument  at time  is estimated as:



For a portfolio, Johnson’s interpolation has the same form as was used in the single financial instrument albeit is different to calculate the first 4 moments of portfolio valuation using matrix expression which requires great deal of calculation. In other words, to estimate the moments of  distribution, covariance matrix, delta, gamma and theta of main variables  should be estimated. Using these we have the possibility to estimate portfolio risk by calculating the relative changes of a portfolio valuation according to the expression on the table[[7]](#footnote-8) above and also by referencing Johnson’s interpolation the same way in the calculation of parameters for the single financial instruments.

Johnson’s interpolation is considered to be the simple method that requires the least numbers of calculation for assessing the portfolio risk with non-linear financial instruments.

* + - 1. **Fourier Transform**

This method is based on the premise that a specified function is disaggregated with Fourier series (the function is approximated or tended to with Taylor series) or is tended to with the use of Fourier series. Despite Fourier series is in general has a geometric form, a complex method is used for risk calculation. If  is the density function of  then Fourier transformation of this function takes the following form:

,  (Here is noted as  for short.) In other words, maneuvering the Fourier transformed function and its specified functions the unique solutions may be found. Thus, in order to find the density function of , its Fourier transform need to be assessed.

Since the Fourier transformation is the complex form of moment generation function, by finding the moment generation function (or the characteristic function), we can find the fourier transform function. In other words, instead of complex conjugate *iu* in the Fourier transform, a moment generation function emerges as to find the real variable *u*.

For single non-linear financial instrument, a relative change in valuation is dependent on change in main variables by  (hereafter, for simplification we not  as  and as  respectively), then the fourier transform takes the following form.



From here the density function of  is the inverse transform of Fourier or can be assessed as follows:



If a portfolio consists of non-linear financial instruments, the relative change of a portfolio would be  whereas the matrix version of it would be . Thus the characteristic function of is defined as follows:

 (Where -the identity matrix  is a covariance matrix). Based on this, a density function of a portfolio has the following form:



* + 1. **Monte Carlo simulation**

Aside from applying the abovementioned or other analytical methods to estimate the market risk for non-linear financial instruments, a risk is estimated alternatively with a Monte Carlo method by composing the sample so as to build the corresponding distribution at that particular time. This method is particularly suitable for a portfolio that consists of both linear and non-linear financial instruments. This method has the advantage of reflecting not only the distribution of a single financial instrument but also that of the entire portfolio. A portfolio risk estimation with the Monte carlo method has the following steps:

1. Drawing samples (or sampling) – Using variances and covariances of main variables of a portfolio, a multidimensional sample of financial instrument valuation is drawn in line with the continuous compounding method via ln.
2. Portfolio valuation – A portfolio is estimated for each member of a sample.
3. Truncating – The outcome from the estimation with the sample, that is a portfolio distribution and the risk of each single financial instrument is estimated.

***Sampling***

The main variables display different properties if a portfolio contains non-linear financial instruments such as options. A stock value and a bond value serve as the main variables for stock and bond option respectively. Thus, to clarify the method, let us take the example of option risk estimation. If the main variables of non-linear financial instruments were stock values, in order to estimate the risk at time *t* (presuming that *0* is the time that risk being estimated) a valuation at time *t* needs to be estimated using stock valuation function. With stock price *P0*, the valuation at time *t* will be . Here  is a standard deviation of the underlying asset (security) at unit time. From here, a value of the underlying is calculated by finding the variable which is standard normally distributed in order to do sampling. For a portfolio, even though the assessment of the standard normally distributed variable is generally the same, the condition that the correlation matrix of randomly generated variables to equal correlation matrix of the main variables  needs to be satisfied.

Although independent and standard normally distributed variables are sampled randomly and directly, a certain transformation needs to be made for with the sampling of correlated or related variables to do the same operation. In other words, a correlation matrix needs to be transformed first in to build the *n* (*n* is the number of financial instruments in a portfolio) standard normally distributed variables with a unit standard deviation for a given  dimensional correlation matrix. To accomplish that, we need to randomly sample *n* independent variables and transform them in such a way that they satisfied the criteria of  dimensional correlation matrix as shown below. Of which:

* Using Cholesky decomposition find the lower triangular matrix that satisfies . For instance, with two random variables the decomposition is done as follows: In line with Cholesky decomposition in, matrix *A* that satisfies  is calculated as follows:

 , 

Here a correlation matrix has to have a positive determinant in order for the Cholesky decomposition worked. However, a determinant of the correlation matrix of the main variables does not have to be positive and in that case other transformation methods can be applied.

*  dimensional, *L* matrix containing independent, random and standard normally distributed variables need to be created.
* Matrix is built via . Thus, each element of has unit variance where the correlation matrix would be .

Thus by generating the random variables with covariance matrix - , we can generate the sample with future values of the underlying asset. Since with the current value of the main variable of the financial instrument *i* thatis, a standard deviation for a unit time that is, its future value would be  and thus the corresponding sample at time *t* becomes. In other words to make the sampling, using each randomly generated variable, future value of a financial instrument *i* can be estimated.

# Portfolio valuation

As previously mentioned, a portfolio value is estimated after the sampling. Based on the number of financial instruments in the portfolio, the time required for the estimation, three types of approximations can be applied to simplify the calculation. Of which:

* Full approximation
* Delta approximation
* Delta-Gamma approximation (theta approximation can also be included)

***• Full approximation of portfolio***

Although simple in some ways, this type of approximation requires great number of calculation. For each element of a sample containing the future values of financial instrument *i,* value of each financial instrument is estimated using valuation function (for instance, stock option, Black-Scholes valuation function can be applied). For example, with Monte Carlo simulation, if after t days, the value of a main variable is, then the value of financial instrument *i* at time *t* will be. In other words, a value of a financial instrument *i* will change at time *t* by. From here, a portfolio change or -is calculated using expression (1).

***•*** *Linear approximation of a portfolio*

If a valuation function of a financial instrument, for instance a Black-Scholes valuation function is used fully for individual financial instrument or a portfolio as a whole with the large sample, a greater number of calculation are needed. Thus, it is advantageous to estimate the valuation using approximation from the perspectives of calculation and other aspects. The simplest type of approximation is a delta method where the valuation of a financial instrument is approximated in a linear fashion. In this case where the value of a financial instrument of a particular time is, and a value of a main variable is  , then the value of financial instrument at time *t* will be as follows.

. Where  or is the first order derivative of a financial instrument valuation with respect to the main variable.

From here, a change of financial instrument at time *t* will be . And using the expression (1), a portfolio change or -is estimated.

## **•** Higher order approximation of a portfolio

The abovementioned approximation method of a financial instrument yields larger error if the change in the main variable is large. In other words, this method is suitable for a small change in a main variable. Thus, in case of a larger change an alternative method that reflect this change is needed. In order to do so, the Taylor series up to the second order derivative is used. In this approximation, aside from using the effect of a second order derivative *gamma*, a change in a value of a financial instrument due to a change in time *theta* (a first order derivative of a valuation function with respect to time) can be included to improve approximation. In this case, with the value of financial instrument and that of a main variable  at a particular time, a value of financial instrument at a time *t* will be the sum of abovementioned effects and has the following form:

, .

Where *t* is the length of time that the valuation is estimated, ,  is the first and second order derivatives respectively. From here, a change in a valuation of financial instrument at time *t* becomes:  . And using the expression (1), a portfolio change or -is estimated.

# Risk estimation

A portfolio risk VaR can be estimated at the chosen confidence level after a portfolio valuation for each component of a compiled sample. To do so, the derived results and the relative changes of a portfolio are sorted to which the estimate is calculated that corresponds to the chosen confidence level.

**Two. Cash flow of financial instrument**

Banks are obliged to operate actively on the financial market with goals of improving their profitability by adequately allocating their financial assets and to obtain their positions either in specified segment of the market or the market as a whole.

In that case, the contingent risk level of particular financial instruments might differ depending on the types, terms and volumes of the financial products or instruments. Subsequently, it is inevitable for banks to forecast and manage the contingent risk levels of future cash flow of their various financial instruments held. That is, since banks are obligated to measure their contingent risks in the future, this section comprises the sets of methodologies for banks to adequately predict their future cash flows.

**2.1. Estimation of the cash flows of financial instruments**

The first step of adequately defining (measuring) the overall risk of the portfolio with various financial instruments is to determine the future cash flows of each financial instrument.

The so-called cash flow is defined in terms of their market value which means the cash flow of the certain financial instrument is discounted to its present value from the prices (exchange rate, interest rate) that are set present in the market. In order to do that, the market price of the instruments for instance, yield curve of the newly issued security (i.e. bond) by bank and discount rate of the discounted bond are needed. (Discount basis rate changes proportionally to the future discounted cash flow of the instrument)

Most financial instruments characteristics are defined by their underlying interest rates and the interest rate determines the allocation of cash flows at certain point in time. The cash flow equality method is used frequently to measure various types of risks.

**2.1.1. Estimation of fixed income financial instruments**

**2.1.1.1. Simple coupon bond**

Simple coupon bond is the bond that promises to pay a specified constant rate of interest for a specified time period, with principal to be repaid when the bond matures. Since the coupon of this bond is constant, cash flows of principal as well as coupons are defined directly as the amount of the coupons and principal at particular point in time. That is, present value of the bond cash flow can be written as:  - the present value of the bond is illustrated as the sum of each amount discounted by its respective interest rate and time factors respectively. (Where,  -is the interest rate of the bond to be calculated at the time t, *f* -is the amount of coupon)

**2.1.1.2. Bond with floating interest rate**

Floating interest rate bond is the financial instrument that promises to pay a specified floating rate of interest for a specified time period, and at the end of the maturity the principal plus its last coupon payment is made when the instrument matures. Though this type of bond is calculated upon the floating interest rate, the interest from previous period is used to estimate the amount of the interest since it is paid on a constant frequency. For instance, if the coupon of the bond is paid every half a year, the 6-month LIBOR rate could be used to estimate the interest amount of the bond within the next 6 months.

If the contract of that instrument is done at time *t* and the coupon of this bond is paid on a regular frequency (i.e. semi-annually, quarterly, annually, etc) except the first payment, it is difficult to derive the direct estimation of cash flows of all payment periods of the bond. (Last period cash flow can only be known when the unexpected interest rate is known.) In that case, forward rate is used to determine amount of the payment at period *i*. If  is the forward rate of *i-1* period, and *P* is the principal of the bond, the cash flow of the period *i* will be. (The forward rate of period *i-1* is). Using these expressions, present value of the future cash flows of the bond with floating interest rate can written: . Inserting forward expression above into the formula above results  which represents the cash flows of the floating rate bond in term of their value at time *t*. (From these statement, if *t=0* then the present value of the floating rate bond equals *P*)

**2.1.1.3. Simple interest rate swap contract**

Financial institutions (hereafter referred to as FIs) use interest rate swap contracts in order to prevent to face the risk of deteriorating the cash flows on the interest payments due to the interest rate fluctuation. As a result of entering the swap contract, the FI has a duty to make fixed-rate payment of the notional amount of the swap (or receive the fixed-rate payment) whereas in turn the other counterpart has to make the variable-rate payment (receive the variable rate payment) to the FI.

In order to determine the cash flow of interest rate swap, for FI whose responsible for making variable-rate payment and receiving fixed-rate payment it is appropriate to deem as FI buying fixed-rate bond and at the same time buying variable-rate bond. Thus, as to derive the expression of net cash flow of the fixed and variable-rate bonds, it is simply the difference between the corresponding present values of the cash flows of those bonds: . (If the above case is present to the FI, in reality, FI pays fixed-rate and receives variable-rate interest amount in their nominal terms. Thus, present value of this contract can be written as the and  instead of using net expression.

**2.1.1.4. Swap contract to be realized in the future**

This type of deal is generally similar to the simple interest rate swap with an exception that this swap contract payment is realized in the future. Thus, when estimating the cash flows of this type instrument using the similar method that of in simple swaps the cash flow of variable interest rate would be unable to be uncertain. For financial institution who entered into the swap contract at time *t* and started making interest payments at time *k,* the present value of the fixed-part of this swap is  whereas the cash flow of the variable-part would be  or in its short form.

**2.1.1.5. Forward rate contract**

Forward rate contract is the one type of financial instrument that is commonly used for risk hedging. That is, this instrument sets protection against long (buying of the forward rate contract) and short (selling of the forward rate contract) rate fluctuations that might occur at any time intervals in the future. In this contract which is done at a particular time interval, a financial institution bears an obligation to borrow (lend) specified financial and payment instrument or product for predetermined interest rate in specified time. Where for a financial institution who has entered the long (short) forward contract for a principal amount of *P* and interest rate of *r* which would be realized at time *t* for payments to be donetimes, then the present value of cash flows of this contract would be expressed as same as borrowing (lending) *P* amount of money today (at 0 time) for  time period and  interest rate and subsequently lend (borrow) this money for *t* time period and interest rate.

**2.1.1.6. Futures rate contract**

Cash flows of futures rate contract are estimated similarly to the cash flows of forward rate contract. The only difference is that, mostly this contract is done with the discounted form of the cash flows.

**2.1.2. Foreign Exchange**

Financial instruments are expressed in the balance sheet in terms of the currency of the particular country (for instance, In financial institutions of United States balance sheet is expressed in terms of US dollar whereas in Mongolia it is tugrug.). A risk related to the financial derivatives is not only constrained by the financial instruments expressed in domestic currencies but also for other derivatives which are expressed in foreign currencies. For 2 financial institutions A and B, if they bought, say, bonds of country A, risk that may face institution A is only interest rate risk where for B its not only that but also the foreign exchange risk also matters. Thus, the important thing in estimating the foreign exchange risk is to determine the cash flows of the derivatives prone to exchange risk.

**2.1.2.1. Spot position**

The cash flows of foreign spot position are defined as the position expressed in terms of the spot rate at the time. For instance receivables are expressed as inflows and payables are as outflows.

**2.1.2.2. Foreign forward trading contract**

Foreign forward contract is the contract which promises to exchange the amount of currency with other currency for specified exchange rate. Cash flows of this instrument are defined by using the spot rate and the interest rate relevant to the time period of the contract. (In reality, other costs such as commission fees etc. are often occurred, but for convenience we are not going to include them to the estimation). If a financial institution entered to the forward contract that promises to buy euro with US dollar in time *t*, then the cash flows of this contract is simply the amount specified in the contract which would be exercised at time *t*. It could be understood as the same way as financial institution borrowing US dollar at rate, and lending euros at rate  at time *0.* If *S* is the dollar/euro spot rate (*S=USD/EUR* is denoted if the financial institution’s main currency is US dollar) then the forward rate is expressed as.

* + - 1. **Foreign swap trading contract**

Cash flow of this instrument is defined as the same way as interest rate swap cash flows. That is, one party of the contract receives the amount of fixed-rate interest of the principal and pays to the other party the variable-rate interest amount of the principal. The difference is that, inflows and outflows are paid in 2 different currencies and the principal payment is made at the beginning and the end of the period. The fixed-rate side of this financial instrument is in fact could be likened to the fixed income bond where the variable-rate side is similar to that of variable-rate financial instrument.

* + 1. **Securities**

Cash flows of securities are determined as spot position in terms of the domestic currency. As for securities in foreign currency, in addition to interest rate risk, the foreign exchange risk would also be related.

* + 1. **Commodities**

Commodity cash flow generally has the similar features to the interest rate cash flows. The possible risks associated with this instrument may arise from fluctuations in the financial market (risk of buying commodities today and hold them for certain period of time) and buying and selling of an asset in the future (for instance, forward or supply of a commodities in the next month).

**2.1.4.1. Commodity futures contract**

This instrument gives its buyer or investor the convenience for trading the commodities and settling its price with the manufacturers and provides the methods of shifting their risks. These types of contracts provide the market players the latest updates of information in the financial market. The cash flows of this instrument are similar to that of forward contract.

**2.2. Cash flow transformation**

While cash flows of some financial instruments have few payment frequencies, other financial instruments have more payment frequencies depending on instrument’s nature and characteristics. The number of cash flows in the portfolio with financial instruments of a bank increases as the number of payments as well as the number of financial instruments increases. It is often difficult to do estimation or even if the there was relevant information it becomes problematical to obtain an efficient assessment of market risk due to the ambiguous fluctuations on the important variables and the correlations between them. Thus, in order to estimate the risk of cash flow fluctuations with more simplified methods, it is important to associate the times of cash flows with their relevant variables. We call this method, the method of transforming cash flows into the notional time periods without violating its properties and characteristics.

**2.2.1. Notional period**

Notional periods with given intervals are used when transforming the cash flows. That is, instead of using actual time periods, notional periods are used to allocate the cash flows in such a way that the basic properties are taken into account. This means that cash flows are allocated to its 2 closest time intervals of its actual periods.

For the purpose with simplifying the potential risk estimations, the methods of allocating the cash flows to their respective time periods are based upon 2 basic concepts. Those are as follows: 1/ to remain applying the notional periods that are used to estimate the risks of linear and non-linear financial products. 2/ to pre-estimation of fluctuations of as well as the interrelationships between financial variables (i.e. interest rate) in the certain notional period, using the latest market data

Following 3 conditions are sustained in the cash flows in notional periods in order to transform the cash flows:

1. **Maintenance of the market value:** The market value of transformed cash flows should equal the market value of its actual cash flows.
2. **Maintenance of market risk:**  The market risk of transformed cash flows should equal the market risk of its actual cash flows.
3. **Maintenance of the properties:** The property of transformed cash flows should equal the property of its actual cash flows. (That is, the properties of inflow and outflows of cash should be the same)

When measuring the potential risks, though it is required to do the transformation into all possible cash flows, it is practically the notional and actual periods are the same when the right notional period-points are chosen.

Following 2 general rules are used for transforming of the cash flows:

1. **Maintenance of present value:**  This means, if actual cash flows are set to occur at time say, *t*, and those cash flows are allocated to the notional periods *t1* and *t2*(), then the present value of cash flows at time *t*, *t1* and *t2* should equal.
2. **Maintenance of duration:** This means that the duration of the cash flows at time *t1* and *t2* ()should be equal to that of at time *t*.

**2.2.2. Allocation of cash flows into notional periods.**

Standard deviation-based method is the one of the methods of allocating the financial instrument’s cash flows into the notional periods. The asset of applying this method to allocate cash flows is that the deviation is used to express the potential risk (VaR estimates using simple or delta method) of a given instrument. In order to simplify the process of estimating loss using transformed cash flows method, the numerical estimates of standard deviations and correlations of various financial instruments that are actively traded in the financial market have to be incorporated into the certain dataset.

Following ordered processes have to be done to allocate the actual cash flows at time *t* into the notional periods *t1* and *t2* (). Where  is weight coefficient of which the actual cash flow is to be allocated at time,  is weight coefficient of which the actual cash flow is to be allocated at time,  is the standard deviation of a discounted bond at time , is the standard deviation of a discounted bond at time ,  is the interest rate of a discounted bond at time , and is the interest rate of a discounted bond at time .

1. *Estimation of yields of basic cash flows:*  Returns of financial instrument at time *t* is estimated as the linear functions of returns at notional periods - *t1* and *t2*. That is,  where.
2. *Estimation of present value of actual cash flows:*  As previously stated,  or the present value of actual cash flows are estimated using the return at time *t1* or.
3. *Estimation of standard deviations of a relative change in basic variables of actual cash flows:* It is worth noting that, the standard deviation at time  can be expressed as the linear functions of respective standard deviations at times *t1* and *t2.* That is, the standard deviation at time is derived from the function of standard deviations at times *t1* and *t2*each weighted by the respective time factors:, where.
4. *Estimation of weights which are used to allocate the actual cash flows:* From  we can derive the following expression. Since  is the only unknown we can manipulate this expression as. Solution of this equation is, where, , . There are two possible outcomes of  to this equation, but for the actual cash flow allocation, only outcome of  which satisfies the above specified conditions would be applied.
5. *Allocation of actual cash flows into the notional periods:* The selected outcome as stated above is the weight coefficient to allocate the actual cash flows, that is, the present value of cash flows at time is weighted by  whereas the present value of cash flows at time is multiplied by  to partition the actual cash flows into the notional periods.

**Annex**

Although in part I, we got acquainted with general risk estimation methods, it did not show the mathematical concepts behind them. This section comprises of broad knowledge of maximum likelihood method, function characteristics, some matrix properties and Cholesky decomposition method. Of all the methods stated above, Cholesky decomposition has an advantage of converting risk estimation methods from simple analytical forms to matrix forms or visa-versa. Maximum likelihood method, based on parametric estimation of risk assessing, is used for single financial instrument or portfolio estimation of risks. Comprehensive details on derivation of the Fourier transformation are provided in this section since the first section only had the brief introduction on transformation to the distribution function of a given variable.

**Annex 1: Cholesky decomposition**

In order to derive the solution to a linear system of equations, the transformation methods in the matrix that composed of coefficients of unknowns in the system are often applied. Some widely used methods are LU and Cholesky transform. The objective to these methods is to transform the matrix in such a way that solutions to the system are derived most conveniently but also not distorting its basic properties. For instance, in order to solve the linear equation, say matrix *A* is decomposed into the product of 2 triangular matrix , *L* the lower triangular and *U* the upper triangular matrix that is . Thus, the initial equation  changes its form to, and this new equation is solved first by solving  and then by solving the remaining part. That is, the initial problem shifts to a simple decomposition of matrix *A* into product of lower and upper triangular matrices.

Covariance (correlation) matrix should be transformed to assess the potential risk for the matrix to be symmetric. If the target matrix is symmetric more simple method such as Cholesky is used rather than its more technical counterpart LU method. In particular, Cholesky and LU methods are similar with the difference that LU method uses the product of lower and upper triangular matrix where Cholesky applies the product of a matrix with its transpose. That is, when the matrix is symmetric, it can be decomposed in  that matrix *L* be a lower triangular matrix. Sometimes this procedure is referred to process of taking square root of matrix *A.* The elements of matrix *L* are expressed as follows:

; 

Monte Carlo method generates *n* draws of *y* random variable in matrix *A* of covariates by creating *n* draws of  random variables of which its correlation is an identity matrix, and multiply each of these independent random variable *y* by transpose of matrix *L* which is a decomposed matrix of a correlation matrix *A.* In other words it means to generate the process as. It can be simply shown that the correlation matrix of *y* is *A.* That is, or.

**Annex 2. Application of transformed matrix**

In financial and economic analysis there arises necessity to explore multifactor relationships and their impacts on a given variable of interest and broad example is the issues of risk estimation. First section of this guideline explored the usage of correlation and covariance matrix that is the interrelationships between variables of interests and in order to simplify the assessment the matrix is converted into a simpler form by obtaining its eigenvalues and eigenvectors. Since eigenvalues and eigenvectors are widely used in financial analysis broadly applied computer software such as MS-EXCEL, Mathcad, Mathematica provide the functions or programs as to estimate them. Thus this section provides broad insights about the eigenvalues and eigenvectors as well as their applications to the risk estimation.

**Eigenvalues and Eigenvectors**

If *A* is the matrix with dimension and the system (is the real number) has the non-zero vector solution *X,* thenis the eigenvalue of matrix *A.* And vector solution *X* associated with  is eigenvector of *A.* The above system of equations can be written as. This expression is often called characteristic equation of *A,* and eigenvalues of *A* is referred to roots (solutions) of the equation. Although this characteristic equation has the obvious solution 0, eigenvalues and vector have to be non-zero. In line with the basic properties of system of linear equations when  then the system should have the non-zero solutions and subsequently is the equation with the unknown and have the same dimension as that of *A.* For each root that is, there should be an associated eigenvectors of *A.*

**Orthogonal matrix**

Two vectors are said to be orthogonal when. *A* is an orthogonal matrix when  and also the symmetric matrix is an orthogonal matrix. Let,  be non-identical eigenvalues of *A* and,  be their associated eigenvectors. (That is  and). First equality can be expressed as   that is. By multiplying both sides of the equality with, gives. Also by multiplying both side of  with becomes, and substituting  in  becomes, where  since. Thus, for matrix *A*, the eigenvectors of 2 eigenvalues are orthogonal.

The other widely used property in the matrix transformation is necessary and sufficient condition for matrix *A* with dimension to be orthogonal is that columns of *A* were orthogonally normalized. Applying this condition, when the square matrix had different eigenvalues, then *n* normalized vectors can be generated by dividingeigenvectors by their eigenvectors. The matrix made of these vectors is orthogonal since necessary and sufficient condition for matrix *A* to orthogonal is the columns of the matrix were orthogonally normalized.

**Matrix diagonalization**

Process of risk estimation canbecome more convenient, organized and simple and give the comprehensive knowledge about risk estimation just by diagonalizing the matrix. That is, when studying the impacts of multi-factors, it can be much convenient by positioning each of the factors in the diagonalized form of covariance (correlation in some case) matrix. Matrix *A* to be diagonalized or a matrix taken in the diagonalized form means  (that is), where is the invertible matrix and has to be degenerate. (Where *D* is a diagonal matrix) If columns of *D* were denoted  and columns of  were, then above expression becomes;  and since the elements of these matrices are equal they would have the forms:,, ..., . That is, since the first section showed  was the column matrix or the vectors  would be the eigenvalues of *A.* Respective eigenvectors for these eigenvalues are. Since we assume that, *Q* was non-zero definite vectors in  are not related linearly. From above illustration, we could gather that matrix *Q,* a diagonalizer matrix of *A* is the matrix that was composed of eigenvalues of *A.* In general, necessary and sufficient condition for matrix *A* with  dimension to be diagonalized that it had the *n* of linearly non-related eigenvectors.

When *A* is symmetric, we can always find *n* diagonalizable and linearly independent eigenvectors. Also *n* orthogonal eigenvectors can be found. Thus, by using orthogonal matrix *Q* we can diagonalize *A.* Hence symmetric matrix *A* is also called orthogonal diagonalizable matrix. Matrix *A* is said to be diagonalized when (that is), where *Q* is an orthogonal and satisfies  and subsequently  should be met. Transposing *A* gives  which is also equal to matrix *A.* Thus, if matrix is an orthogonally diagonalizable, it is also a symmetric matrix. Inversely, when matrix is symmetric, it is also an orthogonally diagonalizable. Thus, sufficient and necessary condition for matrix *A* to be orthogonally diagonalizable is that the same matrix was symmetric.

**Annex 3: Fourier transformation**

First section of this guideline explained that relationship of financial instrument from its underlying variable were non-linear and when evaluation function of underlying variable is decomposed by Taylor series it would take the square form. Market risk estimation often assumes that relative ratio of underlying variable is normally distributed and when financial instrument’s value is linearly dependent from its underlying variable, then the value itself is normally distributed. But when relationship no longer linear that is non-linear or composite of linear and square relationship, the distribution of financial instrument’s relative value becomes uncertain. That is, in order to determine the distribution of the variable that is non-linear or dependent upon both linear as well as non-linear factors function so called Gamma is applied.

**Gamma function, **distribution**

Square of a normally distributed random variable is followed by ****** distribution and the probability density function of this distribution is the Gamma function. General expression of Gamma is  and taking partial integral gives. Based on this property we easily see that, , . Function  satisfies the density function property. Substituting , in  gives  or = . Substituting  again in  gives  which is a chi-square distribution.

Relative change of value in Taylor series can be written as the function of relative change of underlying variable, where *r* is assumed to follow normal distribution. From above expression, in order to define the distribution of the distribution of  has to be determined. If  and the event occurring of  has the set, then the occurrence of the event  is in set. Hence the probability of event *A* is  and using  the expression would become. If we abridge the integral part of the last expression it becomes  or. It is the  distribution or.

**Function characteristic**

The basic Fourier transformation or its inverse transformation subsequently shifts to a simple problem of deriving the characteristic of the function and if the characteristics of density function are known the density function can be derived. As for non-linear financial instrument the above expression is  and characteristic equation of variable has to be determined. That is,  has to be found and to that, we assume *r* to have standard normal distribution (in general, it has the normal distribution and in that case it can easily be converted into a standard normal distribution) characteristic equation will have the following form: . For the purpose of substituting we can rewrite above expression as  or  and substituting  gives  and eventually the equation becomes:

 . And since the part within integral of this expression is equal to  then it becomes.

**Fourier transformation**

Vector of relative change in financial instrument’s underlying variable at a unit of time is, and relative change of the portfolio value is  with the distribution  or Gaussian distribution. In that case as stated in first part, delta-gamma approximation of  becomes as follows:



Where,  is matrix,  is  matrix. (It did not include the square multiplier ½ of Taylor series) And  and  are matrices of the first and the second derivatives of  and since  is symmetric its covariance is just like, would be known and constant. In order to obtain the Fourier transformation of the distribution function on *Y*, Cholesky decomposition is applied to shorten. As in the Annex1 *X* can be decomposed as, where *H* is the triangular matrix which satisfies. Thus,  is an independent and standard normal column vector whose dimension is. Although in general, Cholesky decomposition is used to simplify the estimation, in this case estimation is not shortened but the goal is to derive an independent and standard normal estimate. Following this concept  becomes:



Where and. Resulting matrix  is a symmetric matrix. (In some case, due to the short and long position in the portfolio, is not always a positive definite matrix though it is symmetric.) Thus, this matrix has eigenvalues in the range of  and has respective real and orthogonal  eigenvectors. With the purpose of diagonalizing the matrix  with these eigenvectors, matrix  is defined as defined in the Annex 2 and subsequently matrix  becomes as follows:



Where  is a diagonal matrix composed of eigenvalues in matrix. Using this expression  becomes:



Where  and. The  in the equation is also composed of independent and standard normal elements. As a result, the expression becomes:



Where ’s are the elements of matrix , ’s are the eigenvalues of the matrix . Previously we showed that the Fourier transformation of the part  in  is  and since’s are independent variables,  would take the following form:

 or



It is more convenient to express above equation in a matrix form using initial variables  and which can be demonstrated from the following 2 transformations. (That is rebuilding the transformation that was done onwe can express this equation in term of and):

First:



Second:



From above expressions, Fourier transformation of variable *Y* becomes:



The relative change in financial instrument’s value is not always normally distributed depending on the value function of the certain financial instrument. For instance, if  was the density function of , inverse of the Fourier transformation of this function is:



(Where  is denoted as  for simplicity.) That is the initial function that is used to obtain the function resulting from Fourier transformation and the resulting function from Fourier transformation can both be used to obtain one another.

**Annex 3: Maximum Likelihood method**

The probability of an event to occur at time *t* of observation or the maximum likelihood of the function is calculated as follows:



From above equation, the joint likelihood of an event to occur through all sample period is simply calculated as the product likelihoods at their respective time. That is:  and taking logarithm from both sides gives. Where  is the parameter estimate and the objective is to find optimal parameters  that are summed up (in the likelihood function these parameters are contributed as variables) throughout *n* periods that should maximize the log-likelihood function.

In order to estimate portfolio risk using GARCH method, single financial instrument’s GARCH-like parameters have to be derived by imposing maximum likelihood method. Suppose a portfolio is composed of *n* financial instruments and has *m* observations, then the density function of distribution of relative change of portfolio value takes the following form:

. Where  is the determinant of matrix *A.*

From above equation, maximum likelihood function of portfolio is: . To simplify the equation, natural logarithm is taken from both sides which results: . In GARCH method that was discussed above, each parameter  () is dependent on the parameter matrix and parameters that can maximize *lnL* should be obtained. In extended or portfolio GARCH method, the maximum likelihood function of is transformed in order to reduce the technicality and make it simpler.

**Examples**

In this section, some numerical examples on estimation of potential loss as a result of fluctuation in the exchange on financial instrument that is denominated in foreign currency using numerous methods. Some methods such as heteroschedastic and Johnson transform or Monte Carlo simulation, when used to estimate risk, because of its technicalities special advanced risk-assessment software is needed. Thus, we show in this section some numerical examples on the usage of techniques that are relatively simple and require fewer technicalities such as historical simulation and Delta-Normal methods on the latest data of one of actively operated banks in Mongolia. Currently the most of financial instruments held in the banks are more likely prone to a currency risk rather than interest rate and security-price risk. The fact that most of the interest rates are constant prevents banks from facing interest rate risk. Hence the numerical examples that are shown here mainly triggered to estimate the loss of foreign currency denominated financial instrument as a result of the fluctuation in the price of an instrument.

Mostly the foreign liabilities of financial institutions are in USD, CHY, EUR, GBP, CHF, JPY and their quarterly or monthly reports are given as in tern of above currency and the other currencies. Due to the other foreign liabilities except the currencies above are denominated in US dollars, we followed the same practice in our numerical examples. And 388-day data on the underlying variable or the exchange rates are sampled for the examples. (Also foreign currency position report as well as data of underlying variable or exchange rates of banks is attached.)

**Example1. Historical simulation**

This is considered to be simplest method compared to its counterparts and because of its convenience it gives an analyst the comparative advantage of having the certain insight about risk of financial instruments. MS-Excel and other statistical software provide the function that can get the number from the set of values with the corresponding probability input. For instance, the function called PERCENTILE can calculate above problem and as stated in the first section there is no need to calculate the probability estimates using the corresponding values that are in the frequency set of intervals. That is, this operation is done more precisely and accurately by the PERCENTILE function. The below table is the example of common usage of this function.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | **USD/MNT** | **CHY/MNT** | **EUR/MNT** | **GBP/MNT** | **CHF/MNT** | **JPY/MNT** |
| **5% - significance level** | 0.000000000 | -0.000386548 | -0.009963596 | -0.007399493 | -0.009484367 | -0.008929863 |
| **Position** | 858,915,571.80 | 0.00 | 24,189,055.17 | 150,290,398.12 | 231,905,107.49 | 446,237,259.23 |
| **Possible loss** | 0.00 | 0.00 | -241,009.96 | -1,112,072.76 | -2,199,473.06 | -3,984,837.79 |
| **Loss/capital** | 0.00000% | 0.00000% | -0.00375% | -0.01732% | -0.03426% | -0.06207% |
| **Bank capital** | 6,419,533,475.7 |  |  |  |  |  |

The possible shortcoming of this method is that the scale of the underlying variable is in the discretion of the economist of bank. Also, in this example, although the USD position is largest in the book, contingent loss of this position in the next day is equal to zero. This is due to how the criterion of number of observations is set which in this example it is 388 days. The result of the table above changes the following way as we change the number of observations to 290 days.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | **USD/MNT** | **CHY/MNT** | **EUR/MNT** | **GBP/MNT** | **CHF/MNT** | **JPY/MNT** |
| **5% significance level** | -0.000482668 | -0.000855472 | -0.010147327 | -0.007740576 | -0.010845177 | -0.008995577 |
| **Position** | 858,915,571.80 | 0.00 | 24,189,055.17 | 150,290,398.12 | 231,905,107.49 | 446,237,259.23 |
| **Possible loss** | -414,570.95 | 0.00 | -245,454.24 | -1,163,334.25 | -2,515,051.93 | -4,014,161.69 |
| **Loss/capital** | -0.00646% | 0.00000% | -0.00382% | -0.01812% | -0.03918% | -0.06253% |

The above table shows that the possible loss due to the exchange rate change in USD in the next day is 414.6 thousand tugrigs. From above example it can be gathered that the difficulty of this method is to choose the optimal number of observation.

As for portfolio estimation, the potential risk estimation is practically the same as single foreign currency position. In order to derive portfolio estimation, relative change of the portfolio value is calculated. For instance, relative change of the portfolio value of the last day can be shown as follows:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | **USD/MNT** | **CHY/MNT** | **EUR/MNT** | **GBP/MNT** | **CHF/MNT** | **JPY/MNT** |
| **Position** | 858,915,571.80 | 0.00 | 24,189,055.17 | 150,290,398.12 | 231,905,107.49 | 446,237,259.23 |
| **Share in Portfolio** | 50.2% | 0.0% | 1.4% | 8.8% | 13.5% | 26.1% |
| **Relative change of the underlying variable** | 0.00000000 | -0.00002416 | -0.00689207 | 0.00224100 | -0.00081400 | -0.00125214 |
| **Relative change of portfolio** | -0.00033738 |  |  |  |  |  |

The relative change in the portfolio is calculated as the weighted average of relative change of each financial instrument in the portfolio by its corresponding weights and in this example the result is or.

By following above procedures relative change in portfolio value of each point of the observation can be derived. Sample values of relative change in the portfolio can be calculated with PERCENTILE function of MS-Excel where the result in this example is *-0.00333*. This estimate is the maximum possible change of a portfolio at the 5% of significance level and subsequently the portfolio risk is calculated as (value of the portfolio is assumed to be 1.711.537.391,81 tugrigs) .

**Example2. Delta-Normal method**

This method is applied only to the linearly correlated financial instruments and hence this method is not used for option contract. The table below shows the bank’s foreign currency positions and also the variance of the exchange rate of USD against MNT in the reporting month.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | **USD** | **CHY** | **EUR** | **GBP** | **CHF** | **JPY** |
| **Position** | 858,915,571.8 | 0.0 | 24,189,055.2 | 150,290,398.1 | 231,905,107.5 | 446,237,259.2 |
| **Deviation\*** | 0.000471445 | 0.000485342 | 0.005570175 | 0.004530282 | 0.006114368 | 0.006049665 |
| **Capital** | 6,419,533,475.7 | | | | | |

\*-The standard deviation of exchange rate of USD against MNT

The standard deviation of relative change in USD/MNT in this example is *0.00047* and since financial instruments are assumed to have linear relationship the maximum fluctuation of relative change in USD/MNT, at 5% significance level is. That is, the bank in the next day, may have the loss of *672.185,9* tugrig at most at 5% significance level from its  tugrigs position of USD. Similarly, risk for each foreign currency can be calculated. The table below shows the results for each position:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | **USD** | **CHY** | **EUR** | **GBP** | **CHF** | **JPY** |
| **Position** | 858,915,571.8 | 0.0 | 24,189,055.2 | 150,290,398.1 | 231,905,107.5 | 446,237,259.2 |
| **Deviation** | 0.0004714 | 0.0004853 | 0.0055702 | 0.0045303 | 0.0061144 | 0.0060497 |
| **5%-significance level** | 0.0007826 | 0.0008057 | 0.0092465 | 0.0075203 | 0.0101499 | 0.0100424 |
| **Loss** | 672,185.9 | 0.0 | 222,316.5 | 1,130,224.1 | 2,353,802.1 | 4,481,312.6 |
| **loss/capital** | 0.01047% | 0.00000% | 0.00346% | 0.01761% | 0.03667% | 0.06981% |
| **Bank capital** | 6,419,533,475.7 | | | | | |

Result shows, in the one day horizon, bank at 5% probability, may lose *0.0698* percent of its JPY/MNT position which is the highest among other positions. The aggregate loss that all position may lose within a day as the share of the bank capital is *0.138* percent. But this estimate is reasonable only when the correlation coefficient among cross rates is 0. In reality, say, USD/MNT and EUR/MNT may have correlation. The impact of the correlation may be negative or positive and this impact shall be included in the estimation of losses. In order to do this, covariance matrix of relative changes in all the foreign position against MNT need be computed. Table below shows the result:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | **USD/MNT** | **CHY/MNT** | **EURO/MNT** | **GBP/MNT** | **CHF/MNT** | **JPY/MNT** |
| **USD/MNT** | 0.00000022 | 0.00000022 | 0.00000017 | 0.00000016 | 0.00000030 | 0.00000013 |
| **CHY/MNT** | 0.00000022 | 0.00000024 | 0.00000017 | 0.00000016 | 0.00000031 | 0.00000018 |
| **EURO/MNT** | 0.00000017 | 0.00000017 | 0.00003104 | -0.00001777 | -0.00003149 | -0.00001894 |
| **GBP/MNT** | 0.00000016 | 0.00000016 | -0.00001777 | 0.00002050 | 0.00001892 | 0.00001132 |
| **CHF/MNT** | 0.00000030 | 0.00000031 | -0.00003149 | 0.00001892 | 0.00003739 | 0.00002217 |
| **JPY/MNT** | 0.00000013 | 0.00000018 | -0.00001894 | 0.00001132 | 0.00002217 | 0.00003646 |

With the data above, the possible risk can be computed in two ways. First, the above covariance matrix is multiplied by corresponding value of 5% significance level. Second, covariance matrix is multiplied by total value of the position after multiplying the matrix by both the weights of the components in the portfolio and 5% probability value.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | **USD/MNT** | **CHY/MNT** | **EURO/MNT** | **GBP/MNT** | **CHF/MNT** | **JPY/MNT** |
| **Position** | 858,915,571.8 | 0.0 | 24,189,055.2 | 150,290,398.1 | 231,905,107.5 | 446,237,259.2 |
| **Share in the portfolio** | 50.18% | 0.00% | 1.41% | 8.78% | 13.55% | 26.07% |

The above table can also be expressed as the in terms of the matrix shown in first section of this guideline as follows:





Standard deviation of the portfolio is computed as  and in the example above it is *0.00238925.* The value adjusted in the 5% probability is  which means that in the one day horizon, the total loss that the bank may face due to exchange rate fluctuation is the product of total foreign position and *0.00394226* which is *6.747.317,7* tugrigs in this example*.* The loss to be incurred is the *0.105*% of the bank capital.

**Example 3: Monte-Carlo method**

Monte Carlo method is used to generate randomly the set of random variables followed by the specified function. This method is typically used when the value of financial instrument is non-linearly dependent from its underlying variable and the general example is the option contract. In Mongolian financial market, banks do not tend to enter the derivative agreement with bond and interest rate as its underlying assets, but rather trade foreign currency option contract on a regular basis. But, these financial institutions are entering the financial derivative contract which stipulates between two foreign currencies, not the domestic currency in the one side. Provided that option contract is made for domestic and foreign currency, the risk is computed by multiplying foreign currency by the spot rate. In case where, option contract of two foreign currencies is made, say between EUR and USD, the potential risk of short position is computed as the estimation in linear financial instrument and the risk of option is computed as by first calculating the potential loss of foreign currency instrument and the resulting amount is then multiplied by the spot rate of the currency against MNT.

Below example shows the usage of the Monte-Carlo method where the option contract of two foreign currencies is made. This numerical example shows the option contract that was made in June of this year by one of bank in Mongolia, and the bank sold a call option of buying value of 10 million euros with the EUR/USD rate of 1.234 within 3 months. The spot rate at the time was EUR/USD 1.140. And the standard deviation of the relative change of EUR/USD rate was *0.000333* by using the historical data of the series. The relevant indicators of this deal are shown in the table below:

|  |  |  |  |
| --- | --- | --- | --- |
|  | **Notation** | **Set up** | **Description** |
| Risk free rate of EUR | r | 8% | Input variable |
| Risk free rate of USD | R | 7% | Input variable |
| Standard deviation of EUR/USD | c | 0.1 | To determined given data |
| Spot rate | S | 1.14 | Input variable |
| Strike price | X | 1.234 | Set in the contract |
| Maturity | T | 3 | Set in the contract |
| Standard deviation of relative change in EUR/USD | C | 0.00333 | To determined by given data |

Option valuation function may differ depending on the underlying asset of the option and since the foreign currency option is used in this example Garman-Kohlhagen valuation function shall be used. The function has the following form:



Where





In order to generate the valuation set of the contract, the set of possible EUR/USD series needs to be created. Since the relative changes of series of EUR/USD have standard normal distribution, random draws that follow standard normal distribution is generated in the first place. We created 1014 draws in this example. For the reason that generated random variables would become the probability estimates, we draw normalized value of the corresponding standard normal probability density function by using its inverse function. (The inverse function of standard normal distribution can be estimated by much available computer software. For instance, in MS-Excel the function NORMSINV can be used). We derive the distribution of relative change in EUR/USD by multiplying its initial values by their standard deviation because these estimates are the random variables that are distributed standard normally. From these, we can derive the distribution of EUR/USD and subsequently by using the pre-created set of EUR/USD we can be able to value the option.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **Random draws** | **Variables with standard normal distribution** | **Set of relative change in EUR/USD** | **Set of EUR/USD** | **Option value** |
|  | /1/ | /2/=NORMSINV(/1/) | /3/=/2/x0.00333 | /4/=1.140+  +1.140x/3/ | /5/=Option  Function(/4/) |
| 1 | 0.14676 | -1.05043 | -0.0035 | 1.136009 | 0.093197887 |
| 2 | 0.352361 | -0.37895 | -0.00126 | 1.13856 | 0.091990711 |
| 3 | 0.130213 | -1.12538 | -0.00375 | 1.135724 | 0.093333286 |
| 4 | 0.03677 | -1.78947 | -0.00596 | 1.133201 | 0.094538605 |
| 5 | 0.943539 | 1.585194 | 0.005283 | 1.146023 | 0.088519586 |
| 6 | 0.402393 | -0.24716 | -0.00082 | 1.139061 | 0.091754993 |
| 7 | 0.809942 | 0.877684 | 0.002925 | 1.143335 | 0.089759618 |
| 8 | 0.994276 | 2.528728 | 0.008428 | 1.149608 | 0.086883948 |
| 9 | 0.956733 | 1.713975 | 0.005713 | 1.146512 | 0.088295124 |
| 10 | 0.585543 | 0.216094 | 0.00072 | 1.140821 | 0.090929671 |
| 11 | 0.807066 | 0.867136 | 0.00289 | 1.143295 | 0.089778192 |
| 12 | 0.458864 | -0.1033 | -0.00034 | 1.139608 | 0.091498161 |
| 13 | 0.636633 | 0.349472 | 0.001165 | 1.141328 | 0.090692968 |
| … | … | … | … | … | … |
| 1014 | 0.037046 | -1.78604 | -0.00595 | 1.133214 | 0.094532361 |

The fifth column of the above table set of option values. Thus, we estimate the option values by using the MS-Excel function PERCENTILE at 5% significance level, similar to the Historical Simulation method. This is *0.08841* in our example and at time of estimating risk this estimation becomes *0.09131*. Thus, the bank, within next month at 5% probability might lose  USD per 1 euro. Thus, bank expects to lose 29 thousand USD from 10 million euros and when denominated in MNT it is 34 million tugrigs. In other words, the bank would expect to lose within next month, 34 million tugrigs at most at 5% probability if they decide enter this deal.

**Example4: Cash Flow mapping**

If the cash flows of the financial instrument is occurred in 5.5 months and its underlying variable is announced in 5 and 7 months (Discounted bond etc.) then the risk of this instrument is computed by allocating future cash flows into these notional periods (5 and 7 months). As well as the cash flows of this instrument within 5.5 months are 1000 units, the following information is given:

|  |  |  |
| --- | --- | --- |
|  | **5 months** | **7 months** |
| **Interest** | 6.605% | 6.745% |
| **Standard deviation** | 0.350% | 0.490% |
| **Correlation** | 0.9975 |  |

1. **Estimation of return of actual cash flows:** Based on the returns of the instrument in 5 and 7 months, return in 5.5 months is computed linearly**.** That is, since , the return is .
2. **Estimation of present value of an actual cash flows:** Since cash flows in 5.5 months are 1000, present value of the instrument is: 
3. **Estimation of standard deviation:** The standard deviation in of cash flows in 5.5 months is computed as linear equation from the deviation of cash flows in 5 and 7 months. That is, 
4. **Estimation of weights of actual cash flows:** The variance 5.5-month return is equal to the variance that is the weighted average of the weight coefficients with their corresponding returns in 5 and 7-month notional periods. As stated in the second section, this means the square equation with the unknown. As for this example, it is. The roots of this equation are, . And since ,  will be used.
5. **Allocation of the actual cash flows into the notional periods:**  is the weight coefficient that is used to allocate the actual cash flows to the 7-month notional period and subsequently coefficient for the 5-month notional period is . Thus, cash flows in 5 months are , where the cash flows in 7 months would be .

**BANK OF MONGOLIA**

**SUPERVISION DEPARTMENT**

1. In some case or in the case option, the financial instruments value can be used. [↑](#footnote-ref-2)
2. The new notation pertaining to the specific methodology is put before its meaning. [↑](#footnote-ref-3)
3. Åðºíõèéäºº òóõàéí õýìæèãäýõ¿¿íèé íÿãòûí ôóíêöýýñ õàìààðàí *p* ìàãàäëàëä õàìààðàõ õýìæèãäýõ¿¿íèé àâàõ óòãà [↑](#footnote-ref-4)
4. The method was extracted from the Technical Document by RiskMetrics and it is applied by JPMorgan group. [↑](#footnote-ref-5)
5. Here tr is diagonal operator that takes the sum of diagonal matrix elements [↑](#footnote-ref-6)
6. The Fortran language version of this algorithm can be found in <http://lib.stat.cmu.edu/griffiths-hill/>. [↑](#footnote-ref-7)
7. See annex for the estimation of the central moments of relative changes in portfolio valuation. [↑](#footnote-ref-8)